MULITPLICATIVE THINKING IN RELATION TO COMMUTATIVITY AND FORMS OF REPRESENTATION

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Multiplicative thinking and commutativity in multiplication have proved to be difficult for many children to learn and to use. Different representations of multiplicative situations may enhance the possibilities for children to understand multiplicative properties. In this paper 24 5th grade students' use of multiplicative thinking is related to their understanding of commutativity and their choice of form of representation. The study discloses that students who use multiplicative thinking show understanding of commutativity to a greater extent as well as ability to use multiple forms of representation of multiplication compared to students who use additive thinking.

Key Words: Multiplicative thinking, commutativity, representation of multiplicative situations

Multiplicative thinking is distinctly different from additive thinking even though it is constructed by children from their additive thinking processes (Clark & Kamii, 1996; Vergnaud, 1983). Vergnaud (1983) states that "multiplicative structures rely partly on additive structures; but they also have their own intrinsic organization which is not reducible to additive aspects" (p. 128). One way to describe what distinguishes multiplicative thinking from additive thinking is to consider the different levels of abstraction. In additive thinking there is just one level of abstraction where multiplicative thinking requires a double, or nested, level of abstraction; one must be able to think of both the numbers of objects in each group and the number of groups simultaneously (Clark & Kamii, 1996; Mulligan & Mitchelmore, 2009). The conceptual understanding of multiplication as something more than repeated addition of equal groups is a prerequisite e.g. to the extension of multiplication and division of real numbers (Barmby, Harries, Higgins, & Suggate, 2009), proportional reasoning (Caddle & Brizuela, 2011) and algebra (Gray & Tall, 1994).

The commutative law makes it possible to change the order of numbers in addition, $a + b = b + a$, and multiplication, $a \times b = b \times a$. By using this property, as well as distributivity and associativity, students may develop flexible methods to calculate and develop their understanding of the number system and the arithmetic operations (Young-Loveridge, 2005). Earlier research gives evidence that it is hard for many children to understand commutativity in multiplication and that, in contrast to commutativity in addition; it does not emerge without teaching (Schliemann, Araujo, Cassundé, Macedo, & Nicéas, 1998; Squire, Davies, & Bryant, 2004). However, there is also evidence that students in upper elementary school can use commutativity in multiplication better than they can use distributivity (Squire, et al., 2004).

The term representation has multiple meanings. In this paper forms of representation refer to visual and verbal resources to understand properties of multiplication. Visual representations are used to create opportunities for students to construct mental images of the concept in question and to promote conceptual understanding in opposite to procedural knowledge (Barmby, et al., 2009; Squire, et al., 2004). Visual representations can be concrete objects as well as diagrams which show spatial layout (Young-Loveridge, 2005). Verbal representation can be used for the same reason, by putting the operation into context, a mental image of the operation can be constructed from the story (Nunes & Bryant, 1995). The choice of representation affects whether distributivity and commutativity is used by children to solve problems (Squire, et al., 2004; Young-Loveridge, 2005). However, Barmby et al. (2009) claim that there is not enough research about children's conceptual understanding of multiplication related to different forms of representation.

Taking into consideration the importance of multiplicative thinking, commutativity and forms of representation and the various references to these areas in the literature, the research questions are: 1) Is there a relation between students' use of multiplicative thinking and their understanding of commutativity? 2) Is there a relation between students' use of multiplicative thinking and how they represent multiplication?

SITUATIONS AND FORMS OF REPRESENTATION

Many researchers have made differently detailed classifications of multiplicative real-world situations (e.g. Greer, 1992; Vergnaud, 1983). In this study three situations are used; *equal groups*, *array* and *multiplicative comparison*. Repeated addition of equal groups is a common way to introduce young children to multiplication; e.g. in situations as "there are 5 marbles in each box, there are 3 boxes". The multiplier refers to the number of boxes and the multiplicand to the number of marbles in each box and they are not easily exchanged. Arrays refer to representations of rectangular arrays which can be exemplified by "there are 5 cookies in each row and there are 3 rows of cookies". In this situation the multiplier and the multiplicand are exchangeable; if we change the point of view the array can be discerned as 5 rows of 3 cookies. Multiplicative comparison is when two numbers are compared multiplicatively. An example of a multiplicative comparison which was used in this study is "Max has 150 kr. This is 3 times as much money as Mollie has. How much money has Mollie?" (Question 10).

Multiplicative comparison problems are productive to explore children's ability to use multiplicative thinking (Van Dooren, De Bock, & Verschaffel, 2010). Clark and Kamii (1996) employed a Piagetian task where three fish in increasing sizes are used. The middle-sized fish eats twice as many pieces of food as the smallest fish and the biggest fish three times as many. They found that first graders can use multiplicative thinking but more than half of the fifth graders in their study did not think multiplicatively. Van Dooren et al. (2010) used problems of co-variation, both additive and multiplicative, and they confirmed these findings. They also found that with increasing age students started to overuse multiplicative thinking and the numbers involved in the calculations had a stronger impact on what thinking strategies students used than the structure of the problem. "Easy numbers" with whole number ratios were more often used as cues to use multiplicative thinking, and this was most common among students in fifth grade in their study.

The commutative property is not easily discerned when the form of representation is equal groups. Why are three groups of five exactly as many as five groups of three? (Barmby, et al., 2009; Young-Loveridge, 2005) Young-Loveridge (2005) suggests that rectangular array representations have better potential to visualize commutativity as well as associativity and distributivity. When objects are ordered in rectangular arrays one can see that it does not matter if one thinks of three rows of five or five rows of three. The same fifteen objects can be observed from two directions simultaneously. Evidence from Nunes and Bryant's (1995) study confirms this suggestion; they found that 8-10 year old children more easily could use commutativity in problems that were verbally represented as arrays than as equal groups. This is, however, contradicted by Squire et al. (2004) who found no difference in use of commutativity due to forms of representation. Young-Loveridge (2005) suggests that the array representation of multiplication should be employed alongside other forms representation, in order to "allow students to develop a deeper and more flexible understanding of multiplication/division and to fully appreciate the two-dimensionality of the multiplicative process" (p. 38-39). This is problematized by Barmby et al. (2009) since they found that some children have trouble to discern equal groups in arrays and that those children find array representations abstract. This is also in resonance with findings of Mulligan and Mitchelmore (2009), who found that the ability to discern structure of patterns at age 5-6 years is a predictive factor to 'see' equal groups in rectangular arrays.

METHOD

The study reported here is a part of the author's PhD project, which is a two-year longitudinal study following 27 students through grade 5 to 6. All students in two classes were offered to take part in the study. Out of the 50 students 27 accepted to take full participation. In this part of the study 24 students, 12 girls and 12 boys, in 5^{th} grade participated, since three of them were absent during the part of the data collection were their multiplicative thinking were in focus.

Multiplicative thinking

The students were given a paper-and-pencil test with ten text problems which reflected three different multiplicative situations. Five of the problems reflected equal groups, two rectangular arrays and three multiplicative comparisons. All three types of situations were posed as both multiplication and division problems and the numbers were chosen to be simple to calculate. In order to avoid that the students got tired the test was split into two tests of five questions each and given a week apart. The students could work as long as they wished on the tests, and everyone was finished with each test part within 20 minutes. In the instruction it was explicitly stressed that what was of interest was *how* they work to find the answer, not the answer itself. All students completed the tasks with some kind of explanation of how they reached their answers.

The students' solutions were categorized as additive thinking (AT) or multiplicative thinking (MT). Solutions were considered to show MT when the student wrote solutions involving only multiplication and/or division and to show AT when solutions involved only addition and/or subtraction. Examples of solutions to question 10 (see above) where Erik's answer is coded as AT and Emil's as MT:

Erik: *50 + 50 + 50 = 150. She has 0.*

Emil: *150/3 = 50. She has 50 kr.*

The students were divided into three groups from their solutions to the text problems. Students who showed MT-strategies to his or her solutions in at least 8 of the 10 text problems were categorized into the MT-group. Students who showed AT-strategies in at least 8 of the 10 solutions were categorized into the AT-group and students who showed between 3 and 7 solutions using MT were put into the AT/MT-group.

Commutativity

Students were paired by their teachers. The teachers were asked to put together students that they thought would be able to communicate well together concerning both their social status and their mathematical interest and knowledge. Each student pair was given six questions to discuss and to decide whether the statements given were true or false. Their discussions were videotaped as well as audio recorded and any written material produced by the students was collected. I participated and asked further questions during their discussions. In some pairs it was more like an interview, where I prompted them to explain to each other and to me. In other pairs my role became more of an observer. The session when pairs worked on the tasks were sometimes split into two sessions to keep every occasion to a maximum of 40 minutes. In this paper students' work on task nr 2 is analyzed.

Is it true or false? Explain!

$93 + 56 = 56 + 93$

$93 - 56 = 56 - 93$

$40 \cdot 120 = 120 \cdot 40$

$$\frac{120}{40} = \frac{40}{120}$$

Figure 1. Task nr 2 – commutativity

The students got task 2 (fig. 1) on a card and if necessary it was explained several times that the task was not to calculate but to judge if each of the four expressions was true or false and explain why it was true or false. For each operation additional questions were asked to prompt the students to discuss whether the true/false-statement only applied for specific examples or if it was valid for all numbers. The students' answers to task 2 was categorized into four types of answers; showing confidence of commutativity from start, showing confidence of commutativity after the pair discussion, showing uncertainty whether the commutative law is valid for multiplication, believing that commutativity is not valid for multiplication (table 1).

Table 1. Categories of levels of understanding commutativity

Categories of statements	Examples of students' statements
Confident	*Yes, it is true because it is the same number.*
Confident after discussion	First: *I really don't know.* In the end: *It is really the same number even when you turn it the other way.*
Uncertain	In the context of multiplication facts: *Yes, then it is correct.* In the end: *I am not completely sure. I try to think about ideas of what [multiplication] can be.*
Not commutative	*I might be wrong, but forty times one hundred and twenty is much less than that, because one hundred and twenty times forty becomes a, a fortune.*

Forms of representation for multiplication

Students were asked to orally tell or to write a story to 3 × 9 = 27. Their stories were sorted into stories representing equal groups or arrays. No other situations occurred in the material, but some students wrote stories which did not correspond to 3 × 9= 27.

Table 2. Categories of forms of representation

Representation	Example of story
Equal groups	*Diana has 3 oranges. Each orange has 9 pieces of orange. How many pieces are there?*
Array	*Emma has put all her coins in 3 rows and there are 9 coins in each row. She has 27 coins.* The student also showed with hand movements that the coins were ordered in a rectangular array.
Not corresponding	*Sandra has nine chewing gums. They are on three plates.*

FINDINGS

The analysis of the students' solutions of the ten text problems in the paper and pencil test rendered in three groups of students; 8 who used AT, 9 who used MT and 7 who used AT/MT alternately. The distinction between the MT-group and the other groups was very clear since all students in the MT-group had solved at least 9 of the 10 problems using MT. The students in the AT/MT-group had used MT in 3 to 6 problems. When the students' solutions to the three comparison problems were studied in detail it was found that all eight students in the AT-group showed solutions to two or all three of the comparison problems that were clearly additive and lead to wrong answers. No student who used AT on question 10 (see above) came out with a correct answer; the most common solution was like Erik's: *50 + 50 + 50 = 150. She has 0.*

Among the seven students in the AT/MT-group three students used AT on two of the comparison problems. Among the students in the MT-group all but three students showed multiplicative solutions to all problems. Interestingly all three showed AT solutions to the same multiplicative comparison problem; “Sofia has 50 kr. Martin has 150 kr. How many times more money has Martin?” (Question 8). Two wrote *150 – 50 = 100. 100 more*. The third student showed this solution: *50+50+50 = 150. 3 times more.*

When the students’ discussions about commutativity were analyzed it was obvious that some students were confident from start that addition and multiplication are commutative and subtraction and division are not. They could also explain why in one way or another. Others were not sure about multiplication but during the discussion became certain that multiplication always is commutative. Still others were not sure, or thought it was false when they started to discuss and were still not convinced in the end of the discussion that all multiplication is commutative, even though “it seems like that”. Finally two students who were not sure in the beginning became more convinced that multiplication cannot be commutative if the numbers are really big. The three groups of students who used AT, MT or both AT and MT alternately, were compared to the four levels of comprehension of commutativity. There seems to be a relation between the use of MT and the understanding of commutativity in multiplication (see fig. 2).

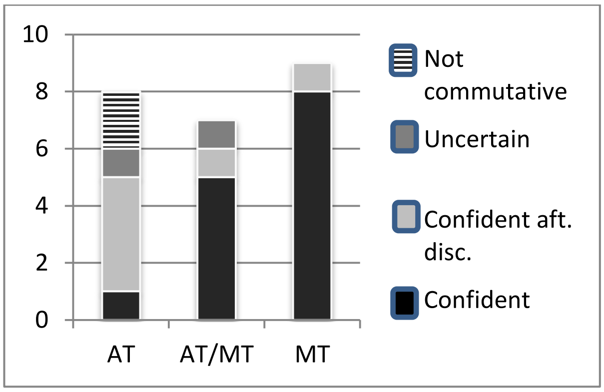

Figure 2: Students views of commutativity according to use of AT/MT.

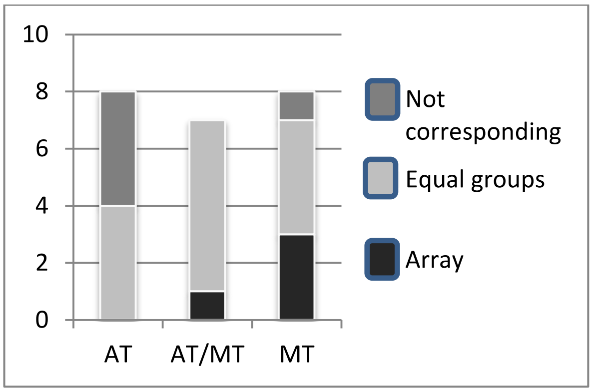

Figure 3: Representation for 3 × 9 = 27 according to use of AT/MT[*]

When the stories produced by the students to illustrate 3 × 9 = 27 were analyzed it was found that four of the students gave stories which represented rectangular arrays. Five students were not able to construct a story to the expression and the remaining 14 students[*] gave examples of equal groups. The three groups of students who used AT, MT or both AT and MT alternately were compared to the representations of multiplications they chose in their stories. These results show that none of the students in the AT-group chose an array representation and half of them did not tell a story that corresponded to the mathematical expression (see fig. 3).

[*] One student was absent during this part of the study.

DISCUSSION

The findings of this study suggest that students' use of multiplicative thinking and their ability to show confidence of commutativity in multiplication are related. The same students who show MT in their solutions to word problems are able to explain commutativity to a greater extent. It is also mainly from this group of students that rectangular array is offered to represent a multiplicative expression, which might imply that MT also is related to multiple forms of representation. The students were however not asked to offer more than one form of representation and it is possible that more students would offer rectangular arrays or other representations if they were explicitly asked to give more examples.

Others (Clark & Kamii, 1996) have found that many students (52%) are not "solid multiplicative thinkers" (p. 48) in fifth grade, which is confirmed in this study. 'Easy numbers' were used in the multiplicative comparisons in this study. Van Dooren et al. (2010) found that 'easy numbers' increased use of MT, even on problems of additive co-variation, especially at this age group. Since no additive problem were posed in this study there is no possibility to compare these findings to theirs, but it is worrying that so few of the students in this study showed MT in their solutions. One reason can be that the students wanted to make sure that their written explanations were easy to follow and showed addition instead of multiplication because they find it clearer, especially with a small multiplier. The fact that all students who used AT on question 10 also came up with wrong answers indicates that MT and the use of multiplicative comparison need to be addressed in teaching at upper elementary grades.

Both MT and commutativity are examples of essential prerequisites to further studies in many areas of mathematics as discussed earlier. If students in the middle grades who exclusively use additive thinking are left to practice multiplication facts, algorithms and other procedures they might not get the opportunity to develop understanding of multiplication as something more than repeated addition of equal groups. To understand the two-dimensional structure of multiplication and commutativity the rectangular array might serve as a better form of representation (Young-Loveridge, 2005). On the other hand Squire et al. (2004) report that the distributive property seems to be easier to understand when it is represented as equal groups. When the findings of this study is considered in comparison to others work on different representations (e.g. Barmby, et al., 2009) it is not possible to draw conclusions of what forms of representation would best serve to enhance the students' possibilities to understand properties of multiplication. The findings, however, do suggest that repeated addition of equal groups is not enough to construct and practice multiplicative thinking and that multiple forms of representation can enhance understanding of commutativity and distributivity, which is supported by other's work (Young-Loveridge, 2005).

The rather small number of students, only 24 in this part of the study, makes it possible to study their development more in-depth. The same students were individually interviewed when they multiplied multi-digit numbers and probed to explain how they carried out the computations. When these data are analyzed they might serve as further evidence on relations between different aspects of students' use of MT. A possible weakness in this study is that the categorization of the students into the three groups using AT and/or MT was made entirely from written work. In contrast, the pair discussions about commutativity were exhaustive and very informative. Since this is the first data collection in a longitudinal study it is probable that data later on might help to understand the development of MT among those students that mainly or exclusively use AT in the first semester of fifth grade.

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